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J. Math. Anal. Appl. 324 (2006) 82–97
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Dynamics of a stochastic Lotka–Volterra model perturbed by white noise

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Received 26 March 2005

Available online 28 December 2005

Submitted by M. Iannelli

Abstract

This paper continues the study of Mao et al. investigating two aspects of the equation

$$dx(t) = \operatorname{diag}\big(x_1(t), \dots, x_n(t)\big)\big[\big(b + Ax(t)\big)\,dt + \sigma x(t)\,dW(t)\big], \quad t \geqslant 0.$$

The first of these is to slightly improve results in [X. Mao, S. Sabais, E. Renshaw, Asymptotic behavior of stochastic Lotka–Volterra model, J. Math. Anal. 287 (2003) 141–156] concerning with the upper-growth rate of the total quantity $\sum_{i=1}^{n} x_i(t)$ of species by weakening hypotheses posed on the coefficients of the equation. The second aspect is to investigate the lower-growth rate of the positive solutions. By using Lyapunov function technique and using a changing time method, we prove that the total quantity $\sum_{i=1}^{n} x_i(t)$ always visits any neighborhood of the point 0 and we simultaneously give estimates for this lower-growth rate.

Keywords: Lotka–Volterra model; Brownian motion; Stochastic differential equation; Asymptotic behavior

1. Introduction

We consider a population consisting of n species. Suppose that the quantity of ith-species at time t is $x_i(t)$ and these quantities satisfy the Lotka–Volterra equation

$$dx(t) = \operatorname{diag}\big(x_1(t), x_2(t), \dots, x_n(t)\big)\big[b + Ax(t)\big]\,dt, \quad t \geqslant 0. \tag{1.1}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.11.064

The details of the ecological significance of such a system are discussed in [2,6].

Without any further hypothesis on the vector b and the matrix A, solutions of (1.1) may not exist on $[0,\infty)$ (see [7] for example). The situation is not better when the population develops under random environment where random factors, being white noise, make influences only on the intrinsic growth rate b, i.e.,

$$dx(t) = \operatorname{diag}\big(x_1(t), x_2(t), \ldots, x_n(t)\big)\big[b + Ax(t) + c\,dW_t\big]\,dt, \quad t \geqslant 0. \tag{1.2}$$

It is easy to give an example to show that solutions of (1.2) may be exploded at a finite time.

Nevertheless, in [7], Mao et al. have shown that if the quantities of population are described by

$$dx(t) = \operatorname{diag}\big(x_1(t), x_2(t), \ldots, x_n(t)\big)\big[b + Ax(t) + \sigma x\,dW_t\big]\,dt, \quad t \geqslant 0, \tag{1.3}$$

i.e., random factor acts on the intraspecific and interspecific coefficients A, then the solution of (1.3), starting from any point $x_0 \in \mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n)\colon x_i > 0,\ 1 \leqslant i \leqslant n\}$ at $t = 0$, exists on $[0,\infty)$. Moreover, authors have estimated the upper-growth rate of the solutions of (1.3) as $t \to \infty$ by using Hypotheses (H1) and (H2) in [7]. The obtained results are very interesting and have a significant meaning in the population theory.

This paper continues the study of Mao et al. investigating two aspects of (1.3). The first of these is to slightly improve results in [7]. To obtain such a result, we need only weaker hypotheses. More concretely, although we keep only Hypothesis (H1), we are able to obtain the same estimates as that obtained in [7].

The second aspect that we shall investigate is the lower-growth rate of the positive solutions which also plays an important role in the population theory as well as in the practice. In many cases, we need to know the extinction rate of the quantities of each species in order to have a suitable policy in investment and to have timely measures to protect them from the extinct disaster. Therefore, in this paper, we are also concerned with the asymptotic behavior of the solution at 0. By using Lyapunov function technique (see [4]) and using a changing time method, we prove that the total quantity $\sum_{i=1}^n x_i(t)$ always visits any neighborhood of the point 0. Further, we are able to give estimates of this convergent rate. It is shown that although $\liminf_{t\to\infty} \sum_{i=1}^n x_i(t) = 0$, this total quantity, from a certain moment t_0, must lie above the curve $y = 1/t^{1+\varepsilon}$, where ε is an arbitrary positive number. On the other hand, the sum $\sum_{i=1}^n x_i(t)$ has to visit fast enough any neighborhood of 0: there are infinitely many times of t such that $\sum_{i=1}^n x_i(t) \leqslant 1/\sqrt{\ln t}$.

The paper is organized as follows: Section 2 deals with a slight improvement of estimates obtained in [7] for the upper-growth rate of the solutions. Section 3 is concerned with a convergent rate of solution to 0. It is proved that the solutions vanish with a rate which is bigger than $1/t^{1+\varepsilon}$ but is smaller than $1/\sqrt{\ln t}$, where ε is an arbitrary positive number.

2. Upper rate estimation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (see [1]). Let $(W(t))_{t\geqslant 0}$ be one-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. We consider the Lotka–Volterra equation perturbed by white noise on the intraspecific and interspecific coefficients A

$$\begin{cases} dx(t) = \operatorname{diag}(x_1(t), x_2(t), \ldots, x_n(t))[b + Ax(t) + \sigma x(t)\,dW(t)], \\ x(0) = x_0 \in \mathbb{R}^n_+, \quad \forall t \geqslant 0, \end{cases} \tag{2.1}$$

where $b \in \mathbb{R}^n_+$ and $\sigma = (\sigma_{ij})_{n\times n}$ is a matrix. Through out of this paper we suppose that:

Hypothesis 2.1.

$$\begin{cases} \sigma_{ii} > 0, & 0 \leqslant i \leqslant n, \\ \sigma_{ij} \geqslant 0, & i \neq j. \end{cases} \tag{H1}$$

The meaning of this hypothesis can be referred to [7].

Theorem 2.2. (See [7, Theorem 5].) *Suppose that* (H1) *holds. Then, there are two constants* α *and* N *such that the following inequality*

$$\limsup_{t\to\infty} \frac{1}{t}\left[\frac{4}{\alpha^2}\ln\sum_{i=1}^n x_i(t) + \int_0^t \left(\sum_{i=1}^n x_i(s)\right)^2 ds\right] \leqslant \frac{4N}{\alpha^2} \tag{2.2}$$

holds with probability 1. *Where,* $x(t)$ *is the solution of* (2.1) *with the initial value* $x(0) = x_0 \in R_+^n$.

Proof. The proof is similar to Theorem 5 in [7]. As is known, the set $\mathbb{R}_+^n$ is invariant, i.e., if $x_0 \in \mathbb{R}_+^n$ then $x(t) \in \mathbb{R}_+^n$ for any $t \geqslant 0$. Denote $S = S(x) = \sum_{i=1}^n x_i$. By applying Ito's formula to the function $V(x) = \ln\sum_{i=1}^n x_i = \ln S(x)$ we obtain

$$dV\big(x(t)\big) = \left[\frac{1}{S}\big(x^\top b + x^\top Ax\big) - \frac{1}{2S^2}\big(x^\top\sigma x\big)^2\right]dt + \frac{1}{S}x^\top\sigma x\, dW(t).$$

Or, equivalently,

$$V\big(x(t)\big) = V\big(x(0)\big) + \int_0^t \left[\frac{1}{S}\big(x^\top b + x^\top Ax\big) - \frac{1}{2S^2}\big(x^\top\sigma x\big)^2\right]ds + M(t), \tag{2.3}$$

where

$$M(t) = \int_0^t \frac{1}{S}x^\top\sigma x\, dW(s)$$

is a real-valued continuous local martingale vanishing at $t = 0$ with quadratic form

$$\big\langle M(t)\big\rangle = \int_0^t \left(\frac{1}{S}x^\top\sigma x\right)^2 ds.$$

Fix an arbitrarily ε $(0 < \varepsilon < 1)$. For any $k \geqslant 1$, an application of the exponential martingale inequality (see [6, Theorem 1.7.4]) gives

$$P\left\{\sup_{0\leqslant t\leqslant k}\left[M(t) - \frac{\varepsilon}{4}\big\langle M(t)\big\rangle\right] > \frac{4\ln k}{\varepsilon}\right\} \leqslant \frac{1}{k^2}.$$

By virtue of the Borel–Cantelli lemma, we can find a set $\Omega' \subset \Omega$ with $P(\Omega') = 1$ and for any $\omega \in \Omega'$ there exists $k_0(\omega)$ such that $\forall k \geqslant k_0(\omega)$

$$\sup_{0\leqslant t\leqslant k}\left[M(t) - \frac{\varepsilon}{4}\big\langle M(t)\big\rangle\right] \leqslant \frac{4\ln k}{\varepsilon}.$$

This relation implies

$$M(t) < \frac{\varepsilon}{4}\langle M(t)\rangle + \frac{4\ln k}{\varepsilon} \quad \text{for all } 0 \leqslant t \leqslant k, \tag{2.4}$$

for $\omega \in \Omega'$ and $k \geqslant k_0(\omega)$. Substituting (2.4) into (2.3) we obtain

$$\begin{aligned} &V\big(x(t)\big) + \frac{1}{4}\int_0^t \frac{1}{S^2}\big(x^\top \sigma x\big)^2\, ds \\ &\quad\leqslant V\big(x(0)\big) + \int_0^t \left(\frac{1}{S}\big(x^\top b + x^\top A x\big) - \frac{1-\varepsilon}{4S^2}\big(x^\top \sigma x\big)^2\right) ds + \frac{4\ln k}{\varepsilon}, \end{aligned} \tag{2.5}$$

for $0 \leqslant t \leqslant k$ and for almost ω and $k \geqslant k_0(\omega)$. On the other hand, from Hypothesis (H1) there exists a constant $\alpha > 0$ such that

$$x^\top \sigma x \geqslant \alpha\left(\sum_{i=1}^n x_i\right)^2, \quad \forall x \in \mathbb{R}_+^n. \tag{2.6}$$

From inequality (2.6) it follows that

$$\frac{1}{S^2}\big(x^\top \sigma x\big)^2\, ds \geqslant \frac{1}{S^2}\alpha^2\left(\sum_{i=1}^n x_i\right)^4 ds = \alpha^2\left(\sum_{i=1}^n x_i\right)^2 ds,$$

which implies

$$V\big(x(t)\big) + \frac{1}{4}\int_0^t \frac{1}{S^2}\big(x(s)^\top \sigma x(s)\big)^2\, ds \geqslant V\big(x(t)\big) + \frac{\alpha^2}{4}\int_0^t \left(\sum_{i=1}^n x_i(s)\right)^2 ds.$$

Moreover, there is a positive number β such that $x^\top b \leqslant \beta\sum_{i=1}^n x_i$ and $|x^\top A x| \leqslant \beta(\sum_{i=1}^n x_i)^2$ for any $x \in \mathbb{R}_+^n$. Therefore,

$$\begin{aligned} \frac{1}{S}\big(x^\top b + x^\top A x\big) - \frac{1-\varepsilon}{4S^2}\big(x^\top \sigma x\big)^2 &\leqslant \beta\left(1 + \sum_{i=1}^n x_i\right) - \frac{1-\varepsilon}{4S^2}\alpha^2\left(\sum_{i=1}^n x_i\right)^4 \\ &= \beta\left(1 + \sum_{i=1}^n x_i\right) - \frac{1-\varepsilon}{4}\alpha^2\left(\sum_{i=1}^n x_i\right)^2. \end{aligned}$$

Let $N = \beta + \frac{\beta^2}{(1-\varepsilon)\alpha^2} < \infty$. Then,

$$\beta\left(1 + \sum_{i=1}^n x_i\right) - \frac{1-\varepsilon}{4}\alpha^2\left(\sum_{i=1}^n x_i\right)^2 \leqslant N, \quad \forall x \in \mathbb{R}_+^n.$$

Hence,

$$\begin{aligned} V\big(x(t)\big) + \frac{\alpha^2}{4}\int_0^t \left(\sum_{i=1}^n x_i(s)\right)^2 ds &\leqslant V\big(x(0)\big) + \int_0^t N\, ds + \frac{4\ln k}{\varepsilon} \\ &\leqslant V\big(x(0)\big) + Nt + \frac{4\ln k}{\varepsilon}, \end{aligned}$$

for any $\omega \in \Omega'$, $k \geqslant k_0(\omega)$ and $0 \leqslant t \leqslant k$.

If $k-1 \leqslant t \leqslant k$ with $k \geqslant k_0(\omega)$ then

$$\frac{1}{t}\left[V\big(x(t)\big)+\frac{\alpha^2}{4}\int_0^t\left(\sum_{i=1}^n x_i(s)\right)^2 ds\right] \leqslant N+\frac{1}{t}\left(V\big(x(0)\big)+\frac{4\ln k}{\varepsilon}\right)$$
$$\leqslant N+\frac{1}{t}\left(V\big(x(0)\big)+\frac{4\ln(t+1)}{\varepsilon}\right),$$

which implies that

$$\limsup_{t\to\infty}\frac{1}{t}\left[V\big(x(t)\big)+\frac{\alpha^2}{4}\int_0^t\left(\sum_{i=1}^n x_i(s)\right)^2 ds\right] \leqslant N.$$

Or,

$$\limsup_{t\to\infty}\frac{1}{t}\left[\frac{4}{\alpha^2}\ln\sum_{i=1}^n x_i(t)+\int_0^t\left(\sum_{i=1}^n x_i(s)\right)^2 ds\right] \leqslant \frac{4N}{\alpha^2}.$$

The proof is completed. □

Let us compare Theorem 2.2 with Theorem 5 in [7]. It is easy to see that $n\sum_{i=1}^n x_i^2 \geqslant (\sum_{i=1}^n x_i)^2 \geqslant \sum_{i=1}^n x_i^2$. Therefore, the estimate (2.2) has the same degree of Theorem 5 in [7] meanwhile in the proof of Theorem 2.2 we need only Hypothesis (H1).

Remark 2.3. Let

$$\begin{cases} dx(t)=\operatorname{diag}(x_1(t),x_2(t),\dots,x_n(t))[h^2(x)(b+Ax(t))+h(x)\sigma x(t)\,dW(t)],\\ x(0)=x_0\in\mathbb{R}^n_+, \quad \forall t\geqslant 0,\end{cases} \tag{2.7}$$

where $h(x):\mathbb{R}^n_+\to\mathbb{R}$ with $0<\alpha_1\leqslant|h(x)|\leqslant\alpha_2$ is a continuous function. This equation differs from (2.1) only by the multiplier $h(x)$. Suppose that Hypothesis (H1) holds, then by a similar way as in the proof of Theorem 2.2 we have

$$\limsup_{t\to\infty}\frac{1}{t}\left[\frac{4}{(\alpha\alpha_1)^2}\ln\sum_{i=1}^n x_i(t)+\int_0^t\left(\sum_{i=1}^n x_i(s)\right)^2 ds\right] \leqslant \frac{4N\alpha_2^2}{(\alpha\alpha_1)^2}.$$

Similarly, we can obtain the following result without Hypothesis (H2) in [7].

Theorem 2.4. (See [7, Theorem 6].) *Let the system parameters $b\in R^n_+$, $A\in R^{n\times n}$ and initial value $x_0\in R^n_+$ be given. Suppose that* (H1) *holds. Then, with probability* 1, *we have*:

$$\limsup_{t\to\infty}\frac{\ln\sum_{i=1}^n x_i(t)}{\ln t}\leqslant 1. \tag{2.8}$$

Proof. Define C^2-function: $V(x(t),t)=e^t\ln\sum_{i=1}^n x_i$. By using Ito's formula, we have

$$dV\big(x(t),t\big)=\left[e^t\ln\sum_{i=1}^n x_i+\frac{e^t}{S}\big(x^\top b+x^\top Ax\big)-\frac{e^t}{2S^2}\big(x^\top\sigma x\big)^2\right]dt+\frac{e^t}{S}x^\top\sigma x\,dW(t),$$

i.e.,

$$V\big(x(t),t\big)=V\big(x(0),0\big)+\int_0^t\Bigg[e^s\ln\sum_{i=1}^n x_i(s)+\frac{e^s}{S}\big(x(s)^\top b+x(s)^\top Ax(s)\big)-\frac{e^s}{2S^2}\big(x(s)^\top\sigma x(s)\big)^2\Bigg]ds+M(t),\tag{2.9}$$

where $M(t)=\int_0^t\frac{e^s}{S}x^\top\sigma x\,dW(s)$ is a local martingale with the quadratic form:

$$\big\langle M(t)\big\rangle=\int_0^t\frac{e^{2s}}{S^2}\big(x^\top\sigma x\big)^2\,ds.\tag{2.10}$$

By virtue of the Borel–Cantelli lemma and of the exponential martingale inequality with $0<\varepsilon<1$, $\theta>1$ and $\gamma>0$, for almost $\omega\in\Omega$, there exists $k_0(\omega)$ such that for every $k\geqslant k_0(\omega)$,

$$M(t)\leqslant\frac{\varepsilon e^{-k\gamma}}{2}\big\langle M(t)\big\rangle+\frac{\theta e^{k\gamma}\ln k}{\varepsilon},\quad 0\leqslant t\leqslant\gamma k.\tag{2.11}$$

Combining (2.9) and (2.11), we obtain

$$V\big(x(t),t\big)\leqslant V\big(x(0),0\big)+\int_0^t\Bigg\{e^s\ln\sum_{i=1}^n x_i(s)+\frac{e^s}{S}\big(x(s)^\top b+x(s)^\top Ax(s)\big)-\frac{e^s}{2S^2}\big(x(s)^\top\sigma x(s)\big)^2+\frac{\varepsilon e^{-k\gamma}}{2}\frac{e^{2s}}{S^2}\big(x(s)^\top\sigma x(s)\big)^2\Bigg\}ds+\frac{\theta e^{k\gamma}\ln k}{\varepsilon}.\tag{2.12}$$

It is easy to see that there exists a constant P independent of k such that

$$\ln\sum_{i=1}^n x_i+\beta\left(1+\sum_{i=1}^n x_i\right)-\alpha^2\frac{1-\varepsilon e^{-k\gamma+s}}{2}\left(\sum_{i=1}^n x_i\right)^2\leqslant P,$$

for any $0\leqslant s\leqslant k\gamma$ and $x\in\mathbb{R}_+^n$. Therefore, with the numbers β and α chosen in the proof of Theorem 2.2 we have

$$\begin{aligned}&\ln\sum_{i=1}^n x_i(s)+\frac{1}{S}\big(x(s)^\top b+x^\top(s)Ax(s)\big)-\frac{1-\varepsilon e^{-k\gamma+s}}{2S^2}\big(x^\top(s)\sigma x(s)\big)^2\\&\quad\leqslant\ln\sum_{i=1}^n x_i(s)+\beta\left(1+\sum_{i=1}^n x_i(s)\right)-\alpha^2\frac{1-\varepsilon e^{-k\gamma+s}}{2}\left(\sum_{i=1}^n x_i(s)\right)^2\leqslant P.\end{aligned}\tag{2.13}$$

From (2.12) and (2.13) it follows that

$$\begin{aligned}V\big(x(t),t\big)=e^t\ln\sum_{i=1}^n x_i&\leqslant V\big(x(0),0\big)+\int_0^t e^sP\,ds+\frac{\theta e^{k\gamma}\ln k}{\varepsilon}\\&=V\big(x(0),0\big)+P\big(e^t-1\big)+\frac{\theta e^{k\gamma}\ln k}{\varepsilon},\end{aligned}$$

for any $0 \leqslant t \leqslant k\gamma$. Thus,

$$\ln \sum_{i=1}^{n} x_i(t) \leqslant e^{-t} V\big(x(0), 0\big) + P\big(1 - e^{-t}\big) + e^{-t} \frac{\theta e^{k\gamma} \ln k}{\varepsilon}.$$

If $(k-1)\gamma \leqslant t \leqslant k\gamma$ and $k \geqslant k_0(\omega)$ we have

$$\frac{\ln \sum_{i=1}^{n} x_i(t)}{\ln t} \leqslant \frac{e^{-(k-1)\gamma}}{\ln(k-1)\gamma}\big(V\big(x(0), 0\big) - P\big) + \frac{P}{\ln(k-1)\gamma} + \frac{\theta e^{\gamma} \ln k}{\varepsilon \ln(k-1)\gamma}.$$

Letting $k \to \infty$ we obtain

$$\limsup_{t\to\infty} \frac{\ln \sum_{i=1}^{n} x_i}{\ln t} \leqslant \frac{\theta e^{\gamma}}{\varepsilon}. \tag{2.14}$$

Since (2.14) holds for every $\gamma > 0$, $\varepsilon < 1$ and $\theta > 1$, then by letting $\gamma \to 0$, $\theta \to 1$ and $\varepsilon \to 1$ we have

$$\limsup_{t\to\infty} \frac{\ln \sum_{i=1}^{n} x_i}{\ln t} \leqslant 1.$$

The proof is complete. □

Example 2.5. We illustrate the above results by the following example. Consider Eq. (2.1) with

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}. \tag{2.15}$$

These parameters satisfy the condition (H1) but do not satisfy the condition (H2) in [7].

We compute a numerical solution, generating 10^6 points with the step-size 10^{-5} and the initial condition $(x_1(0), x_2(0)) = (5, 5)$. This numerical solution is displayed in Fig. 1.

Figure 1 suggests that $\limsup_{t\to\infty} \ln(x_1(t) + x_2(t))/\ln t$ may be much smaller than 1. However, so far we are unable to improve the estimate (2.8).

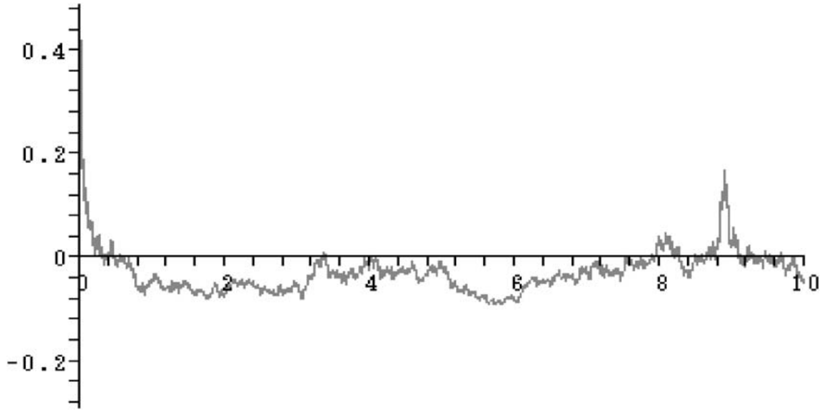

Fig. 1. The graph of the function $y = \ln(x_1(t) + x_2(t))/\ln t$.

Remark 2.6. If $x(t)$ is a solution of Eq. (2.7) with $x(0) \in \mathbb{R}_+^n$ and if Hypothesis (H1) holds, then by the same argument we also obtain

$$\limsup_{t\to\infty} \frac{\ln \sum_{i=1}^n x_i(t)}{\ln t} \leqslant 1. \tag{2.16}$$

3. Asymptotic behavior at 0 as $t \to \infty$

In Section 2 we have studied the upper-growth rate, i.e., $\limsup_{t\to\infty} x(t)$, of the solutions of Eq. (2.1). An estimate of the lower-growth rate, i.e., $\liminf_{t\to\infty} x(t)$ plays an important role in the study of eco-systems because it tells us the rate of the population extinction. First, we consider the one-dimensional case.

3.1. One-dimensional case

Suppose that we have one-dimensional stochastic differential equation:

$$dx(t) = x(t)\bigl(b + ax(t)\bigr)\,dt + g(x)\,dW(t), \tag{3.1}$$

where $b > 0$ and $g(x)$ is a continuous function which satisfies the condition $k_1x^2 < |g(x)| < k_2x^2$. In [3], it has been proved that, with this condition $\limsup_{t\to\infty} x(t) = \infty$ and $\liminf_{t\to\infty} x(t) = 0$ with probability 1. We are now concerned with the rate of this convergence.

Applying Ito's formula to the function $V(x) = 1/x$ we get

$$\begin{aligned} dV\bigl(x(t)\bigr) &= \left(\frac{g^2(x(t))}{x^3(t)} - \frac{x(t)(b + ax(t))}{x^2(t)}\right)dt - \frac{g(x(t))}{x^2(t)}\,dW(t) \\ &= \left(\frac{g^2(x(t))}{x(t)^3} - \frac{b}{x(t)} - a\right)dt - \frac{g(x(t))}{x^2(t)}\,dW(t). \end{aligned} \tag{3.2}$$

Put $y(t) = 1/x(t)$, then Eq. (3.2) can be rewritten as

$$dy(t) = \bigl(-a - by(t) + g^2\bigl(1/y(t)\bigr)y^3(t)\bigr)\,dt - g\bigl(1/y(t)\bigr)y^2(t)\,dW(t). \tag{3.3}$$

We consider a stochastic differential equation:

$$dz(t) = \bigl[-a - bz(t)\bigr]\,dt - g(1/z)z^2(t)\,dW(t), \qquad z(0) = y(0). \tag{3.4}$$

It is easy to see that the solution of Eq. (3.4) exists on $[0, \infty)$ for any $z(0) = y(0) > 0$. On the other hand, $-a - bu < -a - bu + g^2(1/u)u^3$ for any $u > 0$, thus by virtue of the comparison theorem [3, Theorem 1.1, Chapter VI, p. 352], we have

$$z(t) \leqslant y(t), \quad \forall t \geqslant 0. \tag{3.5}$$

On the other hand, we can rewrite (3.4) in the form

$$z(t) = e^{-bt}\left(z_0 - \frac{a}{b}\bigl(e^{bt} - 1\bigr) + M_t\right), \tag{3.6}$$

where

$$M(t) = -\int_0^t e^{bs} g\bigl(1/z(s)\bigr)z^2(s)\,dW(s)$$

is a martingale (note that $k_1 \leqslant |g(1/z)|z^2 \leqslant k_2$). By using law of iterated logarithm we obtain

$$\limsup_{t\to\infty} \frac{M_t}{\sqrt{2\langle M_t\rangle \log\log\langle M_t\rangle}} = 1 \quad \text{a.s.,} \tag{3.7}$$

where $\langle M_t\rangle$ is quadratic form of M_t, i.e.,

$$\langle M_t\rangle = \int_0^t e^{2bs} g^2(1/z) z^4\, ds, \tag{3.8}$$

which satisfies

$$\frac{k_1^2}{2b}\big(e^{2bt}-1\big) < \langle M_t\rangle < \frac{k_2^2}{2b}\big(e^{2bt}-1\big).$$

It is easy to see that

$$\sqrt{2\langle M_t\rangle \log\log\langle M_t\rangle} \geqslant \sqrt{2\frac{k_1^2}{2b}\big(e^{2bt}-1\big)\log\log\frac{k_1^2}{2b}\big(e^{2bt}-1\big)} \sim \frac{k_1}{\sqrt{b}} e^{bt}\sqrt{\log t},$$

as $t \to \infty$. Therefore,

$$\liminf_{t\to\infty} \frac{\sqrt{2\langle M_t\rangle \log\log\langle M_t\rangle}}{e^{bt}\sqrt{\log t}} \geqslant \frac{k_1}{\sqrt{b}}.$$

In order to get more information of this estimate, we need the following lemma.

Lemma 3.1. *Let (x_n) and (y_n) be two sequences of real numbers. If $\limsup_{t\to\infty} x_n > 0$ then*

$$\limsup_{t\to\infty} x_n y_n \geqslant \limsup_{t\to\infty} x_n \liminf_{t\to\infty} y_n.$$

Proof. The proof of this lemma can be deduced directly from the definition of lim sup and lim inf. □

Applying Lemma 3.1 we obtain

$$\limsup_{t\to\infty} \frac{z_t}{\sqrt{\ln t}} = \limsup_{t\to\infty} \frac{e^{-bt} Z_0 - \frac{a}{b} e^{-bt}(e^{bt}-1) + e^{-bt} M_t}{\sqrt{\ln t}} = \limsup_{t\to\infty} \frac{M_t}{e^{bt}\sqrt{\ln t}}$$
$$= \limsup_{t\to\infty} \frac{M_t}{\sqrt{\langle M_t\rangle \log\log\langle M_t\rangle}} \frac{\sqrt{\langle M_t\rangle \log\log\langle M_t\rangle}}{e^{bt}\sqrt{\ln t}} \geqslant \frac{k_1}{\sqrt{b}}.$$

From $y(t) \geqslant z(t)$ it yields

$$\limsup_{t\to\infty} \frac{y(t)}{\sqrt{\ln t}} \geqslant \frac{k_1}{\sqrt{b}}. \tag{3.9}$$

On the other hand, by (2.16) we have

$$\limsup_{t\to\infty} \frac{\ln x(t)}{\ln t} \leqslant 1.$$

This implies that for every $\varepsilon > 0$, there exists $T > 0$ such that

$$\frac{\ln x(t)}{\ln t} \leqslant 1 + \varepsilon \quad \text{for every } t \geqslant T,$$

i.e.,

$$x(t) \leqslant t^{1+\varepsilon} \quad \text{for any } t \geqslant T,$$

which implies that

$$g^2\big(1/y(t)\big)y^3(t) \leqslant k_2^2 t^{1+\varepsilon} \quad \text{for any } t \geqslant T. \tag{3.10}$$

Thus, from (3.3) and (3.10) we get

$$\begin{aligned} y(t) &= e^{-b(t-T)}\left\{y(T) + \int_T^t e^{b(s-T)}\big(\big(-a + g^2(1/y)y^3\big)\,ds + g(1/y)y^2\,dW\big)\right\} \\ &\leqslant e^{-b(t-T)}\left\{y(T) + \int_T^t e^{b(s-T)}\big(\big(-a + k_2^2 s^{1+\varepsilon}\big)\,ds + g(1/y)y^2\,dW\big)\right\}. \end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} \frac{y(t)}{t^{1+\varepsilon}} \leqslant \limsup_{t\to\infty} \frac{k_2^2 e^{-b(t-T)} \int_T^t e^{b(s-T)} s^{1+\varepsilon}\,ds}{t^{1+\varepsilon}} = \frac{k_2^2}{b}.$$

Summing up, we get:

Theorem 3.2. *For any $\varepsilon > 0$, the solution with $x(0) > 0$ of* (3.1) *satisfies the inequalities*

(1) $\displaystyle \limsup_{t\to\infty} \frac{1}{x(t)\sqrt{\ln t}} \geqslant \frac{k_1}{\sqrt{b}}$,

(2) $\displaystyle \limsup_{t\to\infty} \frac{1}{t^{t+\varepsilon}x(t)} \leqslant \frac{k_2^2}{b}$,

with probability 1.

The first inequality in Theorem 3.2 tells us that the population does not extinct too slowly. More exactly, there are a positive constant K_1 and a sequence $t_n \uparrow \infty$ such that $x(t_n) \leqslant K_1/\ln t_n$. Meanwhile, the second inequality says that the population does not extinct too fast, i.e., there are $K_2 > 0$ and $T > 0$ such that $x(t) \geqslant K_2/t^{1+\varepsilon}$ for any $t \geqslant T$.

3.2. Multi-dimensional cases

We now consider Eq. (3.1) in n-dimensional case

$$\begin{cases} dx(t) = \operatorname{diag}(x_1(t), x_2(t), \ldots, x_n(t))[(b + Ax(t))\,dt + \sigma x(t)\,dW(t)], \\ x(0) \in \mathbb{R}_+^n, \quad t \geqslant 0, \end{cases} \tag{3.11}$$

where, $b = (b_1, b_2, \ldots, b_n)^\top \in \mathbb{R}_+^n$ and $A = (a_{ij})$, $\sigma = (\sigma_{ij})$ are $n \times n$ matrices. Suppose that Hypothesis (H1) is satisfied. Put

$$b^i(x) = x_i(t)\left[b_i + \sum_{j=1}^n a_{ij}x_j\right], \qquad \sigma^i(x) = x_i \sum_{k=1}^n \sigma_{ik}x_k.$$

Then (3.11) becomes

$$dx_i(t) = b^i(x)\,dt + \sigma^i(x)\,dW(t). \tag{3.12}$$

Consider

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{n} \sigma^i(x)\sigma^j(x)\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial f}{\partial x_i},$$

the infinitesimal operator of (3.11), defined on the space $C^2(\mathbb{R}_+^n, \mathbb{R})$. Let $p(x) = 1/\sum_{i=1}^n x_i = 1/S(x)$ and $a^{ij}(x) = \sigma^i(x)\sigma^j(x)$. It is easy to see that

$$\begin{aligned} Lp(x) &= \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 p}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial p}{\partial x_i} \\ &= \frac{1}{S^3(x)}\left(x^\top \sigma x\right)^2 - \frac{1}{S^2(x)}\sum_{i=1}^{n} x_i\left(b_i + \sum_{j=1}^{n} a_{ij}x_j\right). \end{aligned}$$

We are going to use the same trick as that given in Section 3.1 to estimate the lower-growth rate of the process $p(x(t))$ where $x(t)$ is a solution of (3.11). In order to do that, we consider the following functions

$$a(x) := \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial p}{\partial x_i}\frac{\partial p}{\partial x_j} = \frac{1}{S^4(x)}\left(x^\top \sigma x\right)^2 > 0, \quad \forall x \in \mathbb{R}_+^n, \tag{3.13}$$

$$\begin{aligned} b(x) &:= \frac{Lp(x)}{a(x)} = \left(S(x) - S^2(x)\sum_{i=1}^{n} x_i\left(b_i + \sum_{j=1}^{n} a_{ij}x_j\right)\right)\Big/\left(x^\top \sigma x\right)^2 \\ &= S(x) - S^2(x)\left(x^\top b + x^\top Ax\right)/\left(x^\top \sigma x\right)^2. \end{aligned} \tag{3.14}$$

Putting $\alpha = \min\{\sigma_{ii}\}$ and $m = \max\{\sigma_{ij}\}$ we have

$$\left(x^\top \sigma x\right)^2 = \left(\sum_{i=1}^{n} x_i \sum_{k=1}^{n} \sigma_{ik}x_k\right)^2 \geqslant \left(\sum_{i=1}^{n} \sigma_{ii}x_i^2\right)^2 \geqslant \frac{\alpha^2}{n^2}S^4(x)$$

and

$$\left(x^\top \sigma x\right)^2 = \left(\sum_{i=1}^{n} x_i \sum_{k=1}^{n} \sigma_{ik}x_k\right)^2 \leqslant m^2 S^4(x).$$

Thus,

$$\frac{\alpha^2}{n^2} \leqslant a(x) \leqslant m^2. \tag{3.15}$$

Let

$$\begin{cases} a^+(\xi) = \sup_{x\in D(\xi,p)} a(x), \\ b^+(\xi) = \sup_{x\in D(\xi,p)} b(x), \end{cases} \qquad \begin{cases} a^-(\xi) = \inf_{x\in D(\xi,p)} a(x), \\ b^-(\xi) = \inf_{x\in D(\xi,p)} b(x), \end{cases}$$

where $D(\xi, p) = \{x \in \mathbb{R}_+^n\colon\ p(x) = \xi\}$ for every $\xi > 0$. It is easy to see that $a^\pm(\xi)$ and $b^\pm(\xi)$ are local Lipschitz positive continuous functions.

Suppose that $\Phi^+(t)$ and $\Phi^-(t)$ are two processes defined by

$$\Phi^+(t) = \int_0^t \frac{a(x_s)}{a^+(p(x_s))}\,ds, \qquad \Phi^-(t) = \int_0^t \frac{a(x_s)}{a^-(p(x_s))}\,ds.$$

It is easy to see that

$$\Phi^+(t) \leqslant t \leqslant \Phi^-(t) \quad \text{for every } t \geqslant 0. \tag{3.16}$$

Suppose that $\psi^+(t)$ (respectively $\psi^-(t)$) is the inverse function of $\Phi^+(t)$ (respectively $\Phi^-(t)$). From (3.16) we see that

$$\psi^-(t) \leqslant t \leqslant \psi^+(t) \quad \text{for every } t \geqslant 0. \tag{3.17}$$

Let $x_t^{(+)} = x(\psi^+(t))$ and $x_t^{(-)} = x(\psi^-(t))$. We see that $x_t^{(+)}$ and $x_t^{(-)}$ satisfy the stochastic differential equations

$$\begin{cases} dx_t^{(+)i} = \left[\frac{a^+(p(x_t^{(+)}))}{a(x_t^{(+)})}\right]^{1/2} \sigma^i(x_t^{(+)})\,dB_t^{(+)} + \left[\frac{a^+(p(x_t^{(+)}))}{a(x_t^{(+)})}\right] b^i(x_t^{(+)})\,dt, \\ x_0^{(+)} = x_0, \quad i = 1, 2, \ldots, n, \end{cases}$$

and

$$\begin{cases} dx_t^{(-)i} = \left[\frac{a^-(p(x_t^{(-)}))}{a(x_t^{(-)})}\right]^{1/2} \sigma^i(x_t^{(-)})\,dB_t^{(-)} + \left[\frac{a^-(p(x_t^{(-)}))}{a(x_t^{(-)})}\right] b^i(x_t^{(-)})\,dt, \\ x_0^{(-)} = x_0, \quad i = 1, 2, \ldots, n, \end{cases}$$

where

$$B_t^{(+)} = \int_0^{\psi^+(t)} \big(a(x_s)/a^+\big(p(x_s)\big)\big)^{1/2}\,dW_t, \qquad B_t^{(-)} = \int_0^{\psi^-(t)} \big(a(x_s)/a^-\big(p(x_s)\big)\big)^{1/2}\,dW_t$$

are two Brownian motions defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ (see [3, Chapter IV, Section 4]). By Ito's formula, we have

$$\begin{aligned} dp\big(x_t^{(+)}\big) = {} & \big[a^+\big(p\big(x_t^{(+)}\big)\big)/a\big(x_t^{(+)}\big)\big]^{1/2} \sum_{i=1}^n \sigma^i\big(x_t^{(+)}\big) \frac{\partial p}{\partial x^i}\big(x_t^{(+)}\big)\,dB_t^{(+)} \\ & + \big[a^+\big(p\big(x_t^{(+)}\big)\big)/a\big(x_t^{(+)}\big)\big](Lp)\big(x_t^{(+)}\big)\,dt. \end{aligned}$$

Since $\frac{\partial p}{\partial x^i} = -p^2$ and $\sum_{i=1}^n \sigma^i(x) = x^\top \sigma x$, it follows that

$$dp\big(x_t^{(+)}\big) = -\big[a^+\big(p\big(x_t^{(+)}\big)\big)\big]^{1/2}\,dB_t^{(+)} + a^+\big(p\big(x_t^{(+)}\big)\big) b\big(x_t^{(+)}\big)\,dt. \tag{3.18}$$

Similarly,

$$\begin{aligned} dp\big(x_t^{(-)}\big) = {} & \big[a^-\big(p\big(x_t^{(-)}\big)\big)/a\big(x_t^{(-)}\big)\big]^{1/2} \sum_{i=1}^n \sigma^i\big(x_t^{(-)}\big) \frac{\partial p}{\partial x^i}\big(x_t^{(-)}\big)\,dB_t^{(-)} \\ & + \big[a^-\big(p\big(x_t^{(-)}\big)\big)/a\big(x_t^{(-)}\big)\big](Lp)\big(x_t^{(-)}\big)\,dt \\ = {} & -\big[a^+\big(p\big(x_t^{(-)}\big)\big)\big]^{1/2}\,dB_t^{(-)} + a^-\big(p\big(x_t^{(-)}\big)\big) b\big(x_t^{(-)}\big)\,dt. \end{aligned} \tag{3.19}$$

From (3.14) and (3.15), it is easy to see that there exist constants $\bar{\alpha} > 0$, $\bar{\beta} > 0$ and $\bar{\gamma} > 0$ such that

$$a^+\big(p(x)\big)b(x) \leqslant -\bar{\alpha}p(x) + \bar{\beta}/p(x) + \bar{\gamma}. \tag{3.20}$$

For any fixed $\varepsilon > 0$, by (2.16) we can find $T > 0$ such that

$$1/p\big(x_t^{(+)}\big) = S\big(x_t^{(+)}\big) \leqslant t^{1+\varepsilon}, \quad \forall t \geqslant T. \tag{3.21}$$

Therefore, (3.18) implies

$$\begin{aligned} e^{\bar{\alpha}t}p\big(x_t^{(+)}\big) &= p\big(x_T^{(+)}\big) - \int_T^t e^{\bar{\alpha}s}\big[a^+\big(p\big(x_s^{(+)}\big)\big)\big]^{1/2}\,dB_s^{(+)} \\ &\quad + \int_T^t e^{\bar{\alpha}s}\big[a^+\big(p\big(x_s^{(+)}\big)\big)b\big(x_s^{(+)}\big) + \bar{\alpha}p\big(x_s^{(+)}\big)\big]\,ds \\ &\leqslant p\big(x_T^{(+)}\big) - \int_T^t e^{\bar{\alpha}s}\big[a^+\big(p\big(x_s^{(+)}\big)\big)\big]^{1/2}\,dB_s^{(+)} + \int_T^t e^{\bar{\alpha}s}\big[\bar{\beta}/p(x) + \bar{\gamma}\big]\,ds \\ &\leqslant p\big(x_T^{(+)}\big) - \int_T^t e^{\bar{\alpha}s}\big[a^+\big(p\big(x_s^{(+)}\big)\big)\big]^{1/2}\,dB_s^{(+)} + \int_T^t e^{\bar{\alpha}s}\big[\bar{\beta}s^{1+\varepsilon} + \bar{\gamma}\big]\,ds. \end{aligned}$$

Since $a^+(\xi)$ is bounded we obtain

$$\limsup_{t\to\infty} \frac{\int_T^t e^{\bar{\alpha}s}[a^+(p(x_s^{(+)}))]^{1/2}\,dB_s^{(+)}}{e^{\bar{\alpha}t}t^{1+\varepsilon}} = 0.$$

Therefore,

$$\limsup_{t\to\infty} \frac{p(x_t^{(+)})}{t^{1+\varepsilon}} \leqslant \frac{\bar{\beta}}{\bar{\alpha}}.$$

By virtue of (3.17), $t \leqslant \psi^+(t)$ for any $t > 0$ it follows that

$$\begin{aligned} \limsup_{t\to\infty} \frac{p(x(t))}{t^{1+\varepsilon}} &= \limsup_{t\to\infty} \frac{p(x(\psi^+(t)))}{\psi^{1+\varepsilon}(t)} = \limsup_{t\to\infty} \frac{p(x(\psi^+(t)))}{t^{1+\varepsilon}} \frac{t^{1+\varepsilon}}{\psi^+(t)^{1+\varepsilon}} \\ &\leqslant \limsup_{t\to\infty} \frac{p(x_t^{(+)})}{t^{1+\varepsilon}} \leqslant \frac{\bar{\beta}}{\bar{\alpha}}. \end{aligned}$$

On the other hand, by (3.14) and (3.15), there exist constants $\underline{\alpha} > 0$ and $\underline{\gamma} > 0$ such that

$$a^-\big(p(x)\big)b(x) \geqslant -\underline{\alpha}p(x) + \underline{\gamma}.$$

Therefore

$$
\begin{aligned}
e^{\underline{\alpha} t} p\big(x_t^{(-)}\big) &= p\big(x_0^{(-)}\big) - \int_0^t e^{\underline{\alpha} s}\big[a^-\big(p\big(x_s^{(-)}\big)\big)\big]^{1/2}\, dB_s^{(-)} \\
&\quad + \int_0^t e^{\underline{\alpha} s}\big[a^+\big(p\big(x_s^{(-)}\big)\big)b\big(x_s^{(-)}\big) + \bar{\alpha}\, p\big(x_s^{(-)}\big)\big]\, ds \\
&\geqslant p\big(x_0^{(-)}\big) - \int_0^t e^{\underline{\alpha} s}\big[a^-\big(p\big(x_s^{(-)}\big)\big)\big]^{1/2}\, dB_s^{(-)} + \int_0^t e^{\underline{\alpha} s}\underline{\gamma}\, ds.
\end{aligned}
$$

Put

$$
m_t = -\int_0^t e^{\underline{\alpha} s}\big[a^-\big(p\big(x_s^{(-)}\big)\big)\big]^{1/2}\, dB_s^{(-)},
$$

where m_t is a martingale with the quadratic form

$$
\langle m_t\rangle = \int_0^t e^{2\underline{\alpha} s} a^-\big(p\big(x_s^{(-)}\big)\big)\, ds.
$$

From (3.15), $a^-(\xi) \geqslant \alpha^2/n^2$ for any $\xi > 0$, and then

$$
\liminf_{t\to\infty} \frac{\sqrt{2\langle m_t\rangle \log\log\langle m_t\rangle}}{e^{\underline{\alpha} t}\sqrt{\ln t}} \geqslant \frac{\alpha}{n\sqrt{\underline{\alpha}}}.
$$

Therefore, by Lemma 3.1 it follows that

$$
\limsup_{t\to\infty} \frac{m_t}{e^{\underline{\alpha} t}\sqrt{\ln t}} = \limsup_{t\to\infty} \frac{m_t}{\sqrt{2\langle m_t\rangle \log\log\langle m_t\rangle}} \frac{\sqrt{2\langle m_t\rangle \log\log\langle m_t\rangle}}{e^{\underline{\alpha} t}\sqrt{\ln t}} \geqslant \frac{\alpha}{n\sqrt{\underline{\alpha}}},
$$

which implies that

$$
\limsup_{t\to\infty} \frac{p(x_t^-)}{\sqrt{\log t}} \geqslant \frac{\alpha}{n\sqrt{\underline{\alpha}}}.
$$

Hence,

$$
\begin{aligned}
\limsup_{t\to\infty} \frac{p(x(t))}{\sqrt{\log t}} &= \limsup_{t\to\infty} \frac{p(x(\psi^-(t)))}{\sqrt{\log \psi^-(t)}} = \limsup_{t\to\infty} \frac{p(x(\psi^-(t)))}{\sqrt{\log t}} \frac{\sqrt{\log t}}{\sqrt{\log \psi^-(t)}} \\
&\geqslant \limsup_{t\to\infty} \frac{p(x(\psi^-(t)))}{\sqrt{\log t}} = \limsup_{t\to\infty} \frac{p(x_t^-)}{\sqrt{\log t}} \geqslant \frac{\alpha}{n\sqrt{\underline{\alpha}}}.
\end{aligned}
$$

This shows that

$$
\limsup_{t\to\infty} \frac{1}{\sqrt{\log t}\,\sum_{i=1}^n x_i(t)} \geqslant \frac{\alpha}{n\sqrt{\underline{\alpha}}}.
$$

Thus, we have proved that:

Theorem 3.3. *Let the system parameters $b \in R_+^n$, $A \in R^{n\times n}$ and initial value $x_0 \in R_+^n$ be given. Suppose that* (H1) *holds. Then, with probability* 1, *there exist two positive constants M, N such that*:

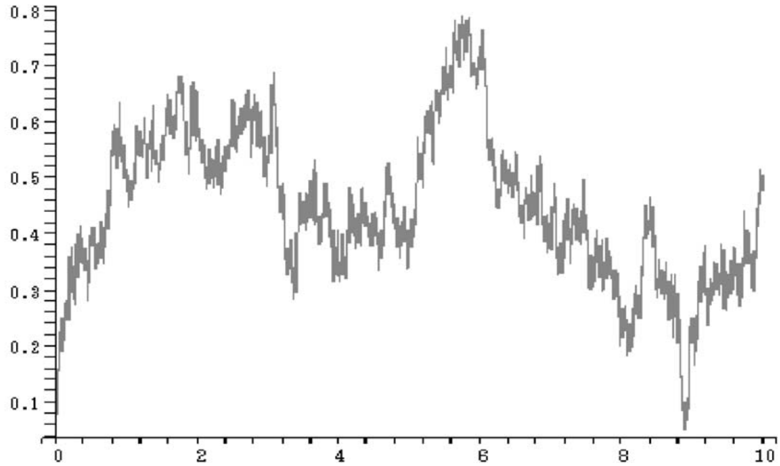

Fig. 2. The graph of the function $y = 1/(\sqrt{\ln t}(x_1(t) + x_2(t)))$.

(1) $\limsup\limits_{t\to\infty} \dfrac{1}{\sqrt{\ln t}\sum_{i=1}^{n} x_i(t)} \geqslant N,$

(2) $\limsup\limits_{t\to\infty} \dfrac{1}{t^{1+\varepsilon}\sum_{i=1}^{n} x_i(t)} \leqslant M.$

Example 3.4. We turn back to the numerical Example 2.5 and show the graph of the function $y = 1/(\sqrt{\ln t}(x_1(t) + x_2(t)))$ in Fig. 2.

From this figure we guess that

$$\limsup_{t\to\infty} \frac{1}{(x_1(t) + x_2(t))\sqrt{\ln t}} > 0,$$

which explains the first inequality of Theorem 3.3.

4. Conclusion

We see that the quantities of population described by a Lotka–Volterra SDE oscillate between 0 and ∞. The upper bound and lower bound of the total quantity $\sum_{i=1}^{n} x_i(t)$ are, respectively, $\theta t^{1+\varepsilon}$ and $\zeta t^{-(1+\varepsilon)}$, where, ε is an arbitrary positive number and θ, ζ are two positive random variables.

We know that an eco-system is perturbed by white noise if it is influenced by many small random factors (see [1,4,5]). On the other hand, when the amount of a species is smaller than a threshold, in fact we consider this species disappears in our system. Therefore, these estimates tell us that for population developing under a random environment, if the white noise makes continually influences on the intraspecific and interspecific coefficients, the total quantity $\sum_{i=1}^{n} x_i(t)$ will be vanished, i.e., all species disappear in eco-system. This conclusion warnes us to have a timely decision to protect species in our eco-system.

Acknowledgments

Authors extend their appreciations to the anonymous referee(s) for his very helpful suggestions which greatly improve this paper.

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